Ternary Quadratic Equation and Harmonic Progression in Rationals

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Abstract – This paper concerns with the problem of obtaining harmonic progression in rational numbers by employing the solutions of the ternary quadratic equation $z^2 = Dx^2 + y^2$. Some examples are also presented.

Index Terms – Ternary quadratic equation, integer solutions, Harmonic progression in rational numbers

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1. INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans)**.** One of the oldest branches of mathematics itself, is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on[1-5,7]. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to Algebra. The theory of Diophantine equations is a treasure house in which the search for many hidden relation and properties among numbers form a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [6,8,9].

This paper concerns with the problem of obtaining harmonic progression in rational numbers by employing the ternary quadratic equation $z^2 = Dx^2 + y^2$. Some numerical examples are also presented.

2. METHOD OF ANALYSIS

The ternary quadratic equation under consideration is

$$z^2 = Dx^2 + y^2, D > 0 \text{ and square free} \quad (1)$$

We illustrate below the process of obtaining triples in Harmonic progression with its elements in rationals by employing (1).

Triple: I

(1) is written as $Dx^2 = z^2 - y^2$

$$\Rightarrow \frac{1}{Dx^2} = \frac{1}{z^2 - y^2}$$

By writing the R.H.S of the above equation is partial fractions, we have

$$\frac{1}{Dx^2} = \frac{1}{2z}\left(\frac{1}{z+y} + \frac{1}{z-y}\right)$$

which is written as $2\left(\frac{z}{x}\right) = \left(\frac{Dx}{z+y}\right) + \left(\frac{Dx}{z-y}\right)$

The above relation implies that the tripe $\{\left(\frac{Dx}{z+y}\right), \frac{z}{x}, \left(\frac{Dx}{z-y}\right)\}$ represents an Arithmetic Progression.

Therefore, by definition, the triple $\{\left(\frac{z+y}{Dx}\right), \frac{x}{z}, \left(\frac{z-y}{Dx}\right)\}$ represents Harmonic progression in rationals

Note that x,y,z in the above triple represent the solutions of the ternary quadratic equation (1). A few examples are given below in table 1.

Table 1: Examples

(x, y,z)	Triple in H.P
$(2rs, Dr^2-s^2, Dr^2+s^2)$	$\left(\frac{r}{s}, \frac{2rs}{Dr^2+s^2}, \frac{s}{Dr}\right)$
$(2rs, r^2-Ds^2, r^2+Ds^2)$	$\left(\frac{r}{Ds}, \frac{2rs}{r^2+Ds^2}, \frac{s}{r}\right)$

Remark:1Observe that the triple $\left(z+y, \frac{Dx^2}{z}, z-y\right)$ also represents Harmonic progression

Triple:II

From (1), we have $\frac{z^2}{x} = \frac{Dx^2+y^2}{x}; \frac{z^2}{y} = \frac{Dx^2+y^2}{y}$

Adding the above two equations and multiplying by 2, we have

$$\left(\frac{2z^2}{x}\right)+\left(\frac{2z^2}{y}\right) = 2\left[\frac{(x+y)(Dx^2+y^2)}{xy}\right]$$

It is seen that the triple $\left(\frac{2z^2}{x}, \frac{(x+y)(Dx^2+y^2)}{xy}, \frac{2z^2}{y}\right)$ forms an Arithmetic progression.

Thus the triple $\left(\frac{x}{2z^2}, \frac{xy}{(x+y)(Dx^2+y^2)}, \frac{y}{2z^2}\right)$ forms Harmonic progression.

A few example are given below in Table 2:

Table 2: Examples

(x, y,z)	Triple in H.P
$(2rs, Dr^2-s^2, Dr^2+s^2)$	$\left(\frac{rs}{(Dr^2+s^2)^2}, \left\{\frac{2rs(Dr^2-s^2)}{(Dr^2+2rs-s^2)} * \frac{1}{(D4r^2s^2+(Dr^2-s^2)^2)}\right\}, \frac{Dr^2-s^2}{2(Dr^2+s^2)^2}\right)$
$(2rs, r^2-Ds^2, r^2+Ds^2)$	$\left(\frac{rs}{(r^2+Ds^2)^2}, \left\{\frac{2rs(r^2-Ds^2)}{(r^2+2rs-Ds^2)} * \frac{1}{4Dr^2s^2+(r^2-Ds^2)^2)}\right\}, \frac{r^2-Ds^2}{2(r^2+Ds^2)^2}\right)$

Remark 2: It is worth to mention that the following triples given by

$$i)\left(\frac{x}{z^2}, \frac{2xy}{(x+y)(Dx^2+y^2)}, \frac{y}{z^2}\right)$$

$$ii)\left(\frac{x}{z}, \frac{2xyz}{(x+y)(Dx^2+y^2)}, \frac{y}{z}\right)$$

$$iii)\left(x(x+y), \frac{2xyz^2}{(Dx^2+y^2)}, y(x+y)\right)$$

$$iv)\left(\frac{x(x+y)}{2z}, \frac{xyz}{(Dx^2+y^2)}, \frac{y(x+y)}{2z}\right)$$

$$v)\left(\frac{x(x+y)}{2}, \frac{xyz^2}{(Dx^2+y^2)}, \frac{y(x+y)}{2}\right)$$

also represent Harmonic progression respectively.

Remark 3: Replacing D by 1 in the above procedure results presented in [10] are obtained.

3. CONCLUSION

In this paper, Harmonic progressions in rationals are obtained after performing some algebra in the ternary quadratic Diophantine equation $z^2 = Dx^2 + y^2$. To conclude, one may attempt to construct progressions through other choices of ternary quadratic Diophantine equations.

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